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Solutions of the Equation of Helmholtz in an Angle IV.

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MATHEMATICS

SOLUTIONS OF THE EQUATION OF HELMHOLTZ IN AN ANGLE *). IV

BY

H. A. LAUWERIER

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1. *Introduction*

This paper is the fourth of a set of papers dealing with the problem of finding solutions of the equation of Helmholtz in an angular region with certain fairly general boundary conditions. The problem may be formulated as follows. The angle A is given in polar coordinates r, φ by $\varphi_1 < \varphi < \varphi_2$. Then a function of Green $G(r, \varphi, r_0, \varphi_0)$ must be determined satisfying the Helmholtz equation

$$(1.1)\quad \left(r^2 \frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r} + \frac{\partial^2}{\partial \varphi^2} - r^2\right) G(r, \varphi, r_0, \varphi_0) = -r_0\, \delta(r-r_0)\, \delta(\varphi-\varphi_0),$$

and the boundary conditions

$$(1.2)\qquad \cos\gamma_j \frac{1}{r}\frac{\partial G}{\partial \varphi} - \sin\gamma_j \frac{\partial G}{\partial r} - \sin\beta_j\, G = 0 \quad \text{for} \quad \varphi = \varphi_j$$

with $j=1$ and $j=2$.

It will be assumed that

$$(1.3)\qquad -\tfrac{1}{2}\pi < \operatorname{Re}\gamma_j \leqq \tfrac{1}{2}\pi, \qquad -\tfrac{1}{2}\pi < \operatorname{Re}\beta_j \leqq \tfrac{1}{2}\pi.$$

This problem differs from that studied in the three preceding papers in the addition of an extra term in the boundary conditions (1.2). In this way the problem formulated above is also a generalization of the well-known problem of the sloping beach where $\gamma_1 = \gamma_2 = 0$ [1]). It will appear below that the generalization does not introduce essentially new features but that the method discussed in the second paper (cf. II section 2) applies equally well in this general case. On the other hand the generalization appears to induce a further simplification of the method.

In order to avoid undue repetition of the arguments a number of mathematical details will be omitted here. We hope that the seeming loss of mathematical rigour will be compensated by a clearer exposition of the method.

The important special case of a half-plane will be investigated in a subsequent paper.

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*) Report TW 70 of the Mathematical Centre, Amsterdam.

[1]) Cf. LAUWERIER (1959a).

2. *The F-problem*

By taking the Fourier transform (see I section 5)

$$P(u, \varphi) = \operatorname{ch} u \int_{-\infty}^{\infty} e^{ir \operatorname{sh} u} F(r, \varphi)\, dr, \tag{2.1}$$

where for $r < 0$ the function $F(r, \varphi)$ is defined in some convenient way, a potential function is obtained which may be represented as the sum of an analytic function of $u + i\varphi$ and one of $u - i\varphi$. By inversion of (2.1) the following representation of the desired solution is obtained.

$$F(r, \varphi) = \int_{-\infty}^{\infty} e^{-ir \operatorname{sh} u} \{f_1(u + i\varphi) + f_2(u - i\varphi)\}\, du, \tag{2.2}$$

where $f_1(w)$ and $f_2(w)$ are analytic functions of which $f_1(w)$ is regular in the strip $\varphi_1 \leqq \operatorname{Im} w \leqq \varphi_2$ and $f_2(w)$ is regular in the strip $-\varphi_2 \leqq \operatorname{Im} w \leqq -\varphi_1$.

The boundary conditions (1.2) are fulfilled if for $j = 1$ and $j = 2$ (cf. I 5.8)

$$\{\operatorname{ch}(u - i\gamma_j) + \sin \beta_j\}\, f_1(u + i\varphi_j) = \{\operatorname{ch}(u + i\gamma_j) - \sin \beta_j\}\, f_2(u - i\varphi_j). \tag{2.3}$$

We shall now introduce an auxiliary function $e(z, \gamma, \beta)$ which satisfies the functional relation

$$\frac{e(z + i\theta, \gamma, \beta)}{e(z - i\theta, \gamma, \beta)} = \frac{\operatorname{ch}(z + i\gamma) - \sin \beta}{\operatorname{ch}(z - i\gamma) + \sin \beta}, \tag{2.4}$$

with $\theta = \varphi_2 - \varphi_1$, and for which $e(0, \gamma, \beta) = 1$.

We note that for $\beta = 0$ the relation (2.4) passes into the form I (4.1) so that $e(z, \gamma, \beta)$ is an obvious generalization of the auxiliary function $e(z, \gamma)$ studied earlier. Assuming the existence of a solution of (2.4) we may put

$$\begin{cases} f_1(w) = \dfrac{e(w - i\varphi_1, \gamma_2, \beta_2)}{e(w - i\varphi_2, \gamma_1, \beta_1)}, \\[2ex] f_2(w) = \dfrac{e(w + i\varphi_1, \gamma_2, \beta_2)}{e(w + i\varphi_2, \gamma_1, \beta_1)}. \end{cases} \tag{2.5}$$

An explicit expression of $e(z, \gamma, \beta)$ may be obtained in a similar way as in I section 4.

Putting tentatively

$$\frac{d}{dz} \ln\, e(z, \gamma, \beta) = \int_{-\infty}^{\infty} e^{-itz} \psi(t)\, dt, \tag{2.6}$$

we obtain by logarithmic differentiation of (2.4)

$$2 \int_{-\infty}^{\infty} e^{-itz} \operatorname{sh} \theta t\, \psi(t)\, dt = \frac{d}{dz} \ln \frac{\operatorname{ch}(z + i\gamma) - \sin \beta}{\operatorname{ch}(z - i\gamma) + \sin \beta}.$$

Inversion gives at once

$$\operatorname{sh} \theta t\, \psi(t) = \frac{1}{4\pi} \int_{-\infty}^{\infty} e^{itz} \left\{ \frac{\operatorname{sh}(z + i\gamma)}{\operatorname{ch}(z + i\gamma) - \sin \beta} - \frac{\operatorname{sh}(z - i\gamma)}{\operatorname{ch}(z - i\gamma) + \sin \beta} \right\} dz. \tag{2.7}$$

The right-hand side of (2.7) can be evaluated by means of the calculus of residues. In order to get a non-ambiguous result we shall restrict the discussion to that of the important subcase

$$|\mathrm{Re}\,\beta|+|\mathrm{Re}\,\gamma|<\tfrac{1}{2}\pi. \tag{2.8}$$

Then we find without difficulty

$$\mathrm{sh}\,\theta t\,\psi(t)=\frac{\mathrm{i}}{\mathrm{sh}\,\pi t}\{\mathrm{sh}\,\tfrac{1}{2}\pi t\,\mathrm{ch}\,\gamma t\,\mathrm{sh}\,\beta t+\mathrm{ch}\,\tfrac{1}{2}\pi t\,\mathrm{sh}\,\gamma t\,\mathrm{ch}\,\beta t\}, \tag{2.9}$$

or written in a somewhat different way

$$\mathrm{sh}\,\theta t\,\psi(t)=\frac{\mathrm{i}}{2\,\mathrm{sh}\,\pi t}\{\mathrm{sh}\,(\tfrac{1}{2}\pi+\gamma)\,t\exp\beta t-\mathrm{sh}\,(\tfrac{1}{2}\pi-\gamma)\,t\exp-\beta t\}. \tag{2.10}$$

Hence we have either

$$\left\{\begin{aligned}\ln e(z,\gamma,\beta)&=\int_0^\infty\frac{1-\cos tz}{t}\,\frac{\mathrm{sh}\,\gamma t\,\mathrm{ch}\,\beta t}{\mathrm{sh}\,\theta t\,\mathrm{sh}\,\tfrac{1}{2}\pi t}\,dt+\\&\quad+\mathrm{i}\int_0^\infty\frac{\sin tz}{t}\,\frac{\mathrm{ch}\,\gamma t\,\mathrm{sh}\,\beta t}{\mathrm{sh}\,\theta t\,\mathrm{ch}\,\tfrac{1}{2}\pi t}\,dt,\end{aligned}\right. \tag{2.11}$$

which is valid for $|\mathrm{Im}\,z|<\theta+\tfrac{1}{2}\pi-|\mathrm{Re}\,\gamma|-|\mathrm{Re}\,\beta|$.
or

$$\left\{\begin{aligned}\ln e(z,\gamma,\beta)&=\int_0^\infty\frac{1-\cos tz_1}{t}\,\frac{\mathrm{sh}\,(\tfrac{1}{2}\pi+\gamma)\,t}{\mathrm{sh}\,\theta t\,\mathrm{sh}\,\pi t}\,dt+\\&\quad-\int_0^\infty\frac{1-\cos tz_2}{t}\,\frac{\mathrm{sh}\,(\tfrac{1}{2}\pi-\gamma)\,t}{\mathrm{sh}\,\theta t\,\mathrm{sh}\,\pi t}\,dt+C,\end{aligned}\right. \tag{2.12}$$

where

$$z_1=z+\mathrm{i}\beta\qquad z_2=z-\mathrm{i}\beta,$$

and where C is a constant which causes the vanishing of the right-hand side of (2.11) for $z=0$.

We note that for $\beta=0$ both expressions (2.11) and (2.12) reduce to the expression I (4.5) of $\ln e(z,\gamma)$. In a similar way as in I section 4 an infinite product can be derived from (2.12). We find, at least formally, with C being some constant

$$e(z,\gamma,\beta)=C\prod\nolimits_1\left\{\frac{z_1^2+(S-\gamma)^2}{z_2^2+(S+\gamma)^2}\right\}\prod\nolimits_2\left\{\frac{z_2^2+(S-\gamma)^2}{z_1^2+(S+\gamma)^2}\right\}, \tag{2.13}$$

where $S=(2m+1)\theta+(2n+1)\tfrac{1}{2}\pi$, $m, n=0, 1, 2, \ldots$, and where the integer n takes even values in $\prod_1$ and odd values in $\prod_2$. The expansion (2.13) can be made convergent obviously by introducing suitable exponential factors. From (2.13) the poles and zeros of $e(z,\gamma,\beta)$ can be deduced by inspection. The first few poles and zeros are

(2.14) poles $\quad z=\mathrm{i}\{\beta-(\tfrac{1}{2}\pi+\gamma+\theta)\}$,
$\quad z=\mathrm{i}\{\beta+(\tfrac{1}{2}\pi+\gamma+\theta)\}$,

(2.15) zeros $\quad z=\mathrm{i}\{-\beta-(\tfrac{1}{2}\pi-\gamma+\theta)\}$,
$\quad z=\mathrm{i}\{-\beta+(\tfrac{1}{2}\pi-\gamma+\theta)\}$.

The situation is sketched below for $\operatorname{Re} \beta > 0$

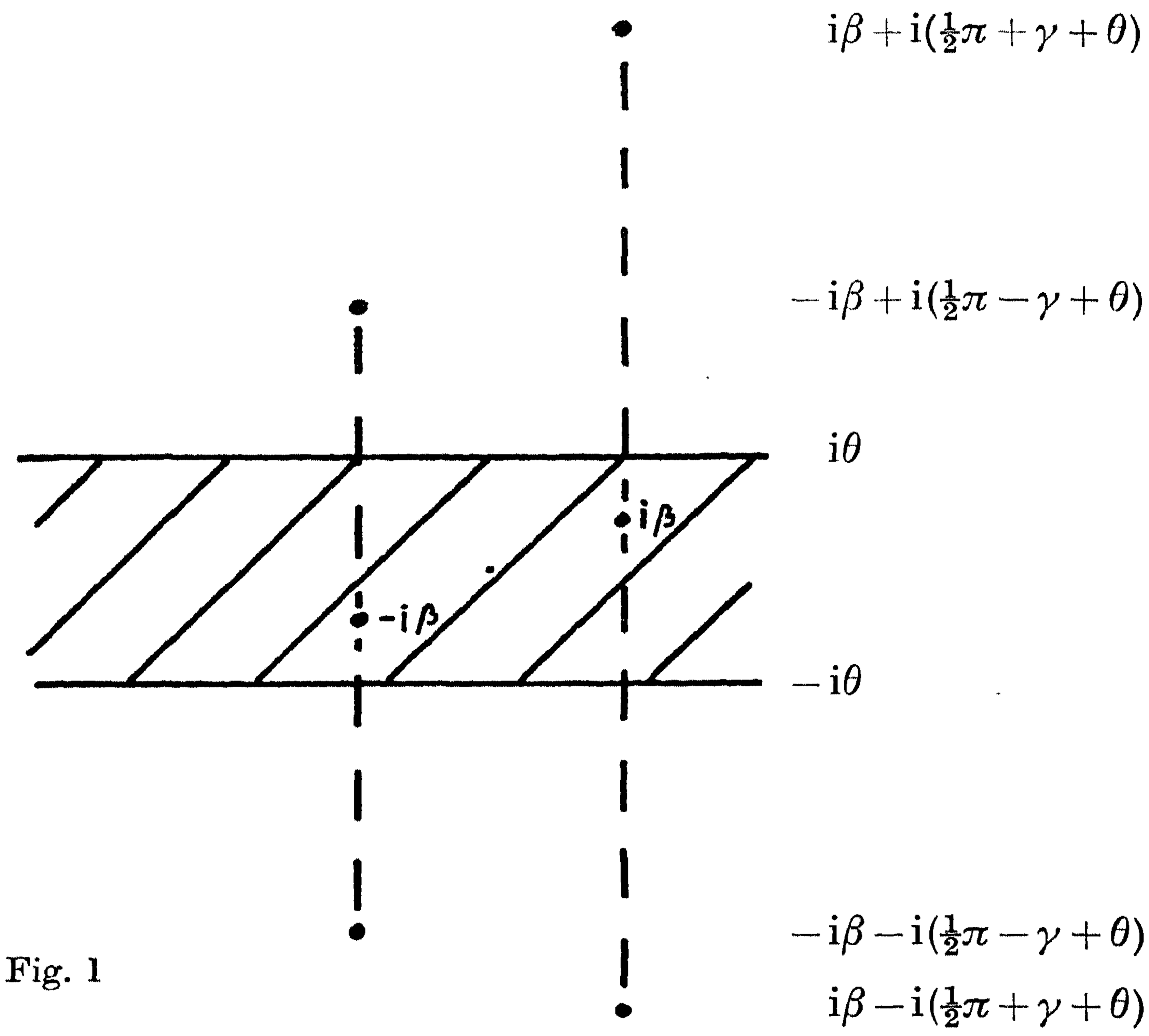

Fig. 1

The asymptotic behaviour of $e(z, \gamma, \beta)$ for $\operatorname{Re} z \to \pm \infty$ can be determined by means of the standard methods e.g. by using (2.11) or in a similar way as in I section 4. We find for $\nu \neq 1$ (cf. I 4.16), $\nu = \pi/\theta$,

$$\ln e(z, \gamma, \beta) = \frac{\gamma}{\theta} \ln \operatorname{ch} z + \text{constant} + O(\exp - \nu|\operatorname{Re} z|) + \\ + O(\exp - |\operatorname{Re} z|). \tag{2.16}$$

Returning to the discussion of the functions $f_1(w)$ and $f_2(w)$ of (2.5) we note that their poles can be derived from the poles and zeros of the e-functions. The first few poles of $f_1(w)$ are schematically given below in figure 2,

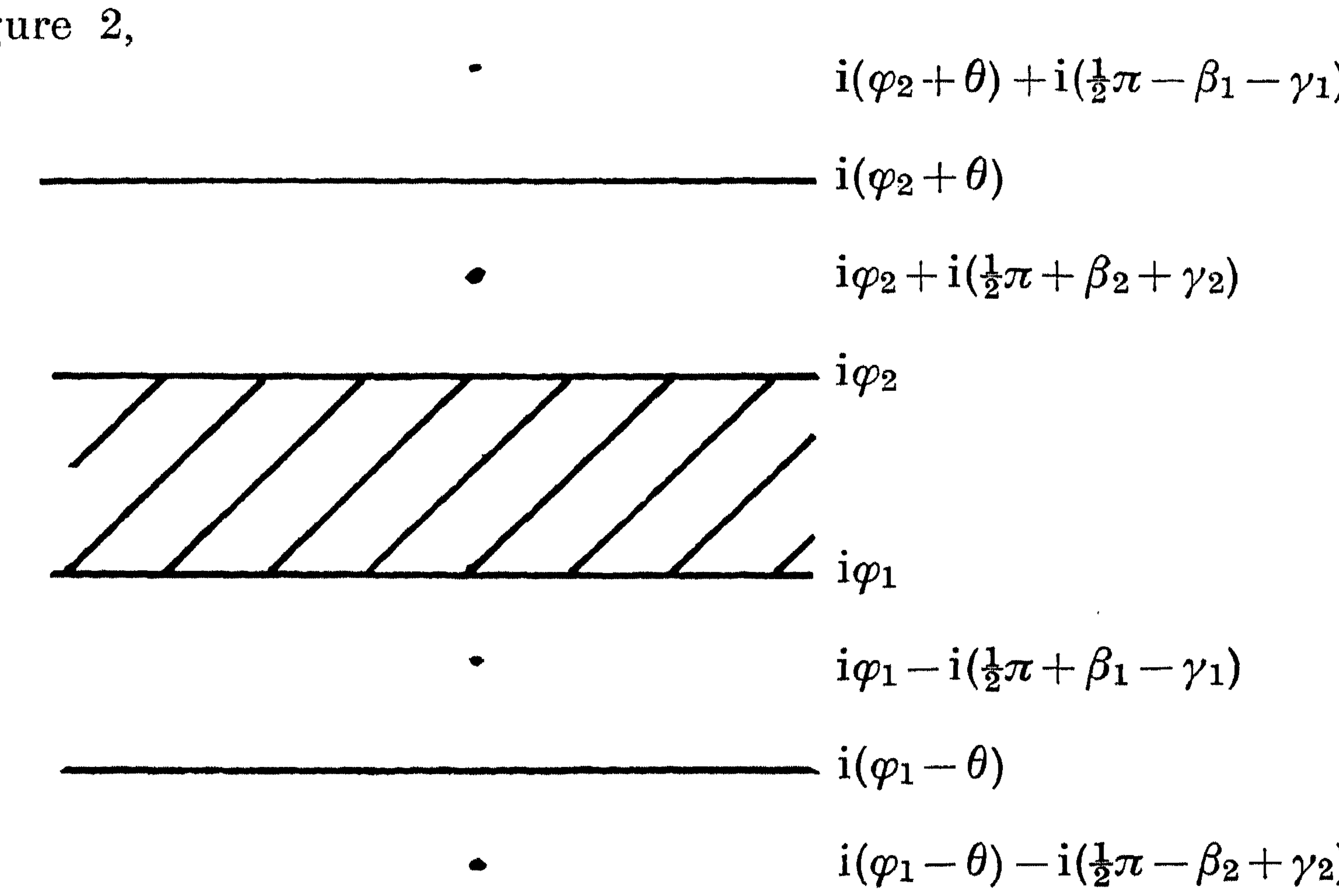

Fig. 2

where we have assumed that α_j, γ_j $(j=1, 2)$ satisfy the inequality (2.8). For $f_2(w)$ a similar picture may be obtained. We see that $f_1(w)$ is free from poles in the strip $\varphi_1 < \operatorname{Im} w < \varphi_2$ and that $f_2(w)$ is free from poles in the corresponding strip $-\varphi_2 < \operatorname{Im} w < -\varphi_1$. Their asymptotic expansion follows at once from (2.16). For both $j=1$ and $j=2$ we have obviously for $|\operatorname{Re} w| \to \infty$

$$\ln f_j(w) = \frac{\gamma_2 - \gamma_1}{\theta} \ln \operatorname{ch} w + \mathrm{O}(1). \tag{2.17}$$

Hence we arrive at a similar conclusion as in the simpler problem treated in I section 5.

The solution (2.2) with f_1 and f_2 given by (2.5) is regular in the sector $\varphi_1 \leqq \varphi \leqq \varphi_2$, $0 < r < \infty$ with the possible exception of the vertex $r=0$. For $r \to 0$ we have (cf. I 5.13)

$$\left\{\begin{array}{ll} \operatorname{Re} \gamma_1 > \operatorname{Re} \gamma_2 & F(r, \varphi) \text{ finite,} \\ \operatorname{Re} \gamma_1 = \operatorname{Re} \gamma_2 & F(r, \varphi) = C \ln r + \mathrm{O}(1), \\ \operatorname{Re} \gamma_1 < \operatorname{Re} \gamma_2 & F(r, \varphi) = C \exp \left\{ \dfrac{\gamma_2 - \gamma_2}{\theta} \ln r \right\} [1 + \mathrm{o}(1)], \end{array}\right. \tag{2.18}$$

where C is some constant.

To $f_1(w)$ and $f_2(w)$ appearing in the right-hand side of (2.2) a regular periodic function $P(w)$ with the period $2\theta \mathrm{i}$ may be added. It is clear that this has no effect on the relations (2.3). In this way a set of higher solutions is obtained viz. (cf. I 5.11)

$$\left\{\begin{array}{l} F_m(r, \varphi) = \\ = \int\limits_{-\infty}^{\infty} \mathrm{e}^{-r \operatorname{sh} u} \{ f_1(u + \mathrm{i}\varphi) \operatorname{ch} m\nu(u + \mathrm{i}\varphi) + f_2(u - \mathrm{i}\varphi) \operatorname{ch} m\nu(u - \mathrm{i}\varphi) \} \, du, \end{array}\right. \tag{2.19}$$

where $m = 0, 1, 2, \ldots$.

In view of (2.17) the behaviour of $F_m(r, \varphi)$ at $r=0$ is of the following kind

$$F_m(r, \varphi) \approx r^{((\gamma_2 - \gamma_1)/\theta) - m\nu}. \tag{2.20}$$

In a similar way as in the first paper (cf. I 5.24) we may construct a second set of solutions of the homogeneous problem by way of

$$\left\{\begin{array}{l} F_m{}^*(r, \varphi) = \dfrac{1}{2\pi \mathrm{i}} \displaystyle\int_L \mathrm{e}^{-\mathrm{i} r \operatorname{sh} w} \{ f_1(w + \mathrm{i}\varphi) \mathrm{e}^{-\sigma m\nu(w + \mathrm{i}\varphi)} + \\ \qquad\qquad + f_2(w - \mathrm{i}\varphi) \mathrm{e}^{-\sigma m\nu(w - \mathrm{i}\varphi)} \} \, dw, \end{array}\right. \tag{2.21}$$

where $\sigma = \operatorname{sgn} \operatorname{Re} w$, and where the contour L is shown below and is assumed to be sufficiently far away that the poles of f_1 and f_2 are outside L.

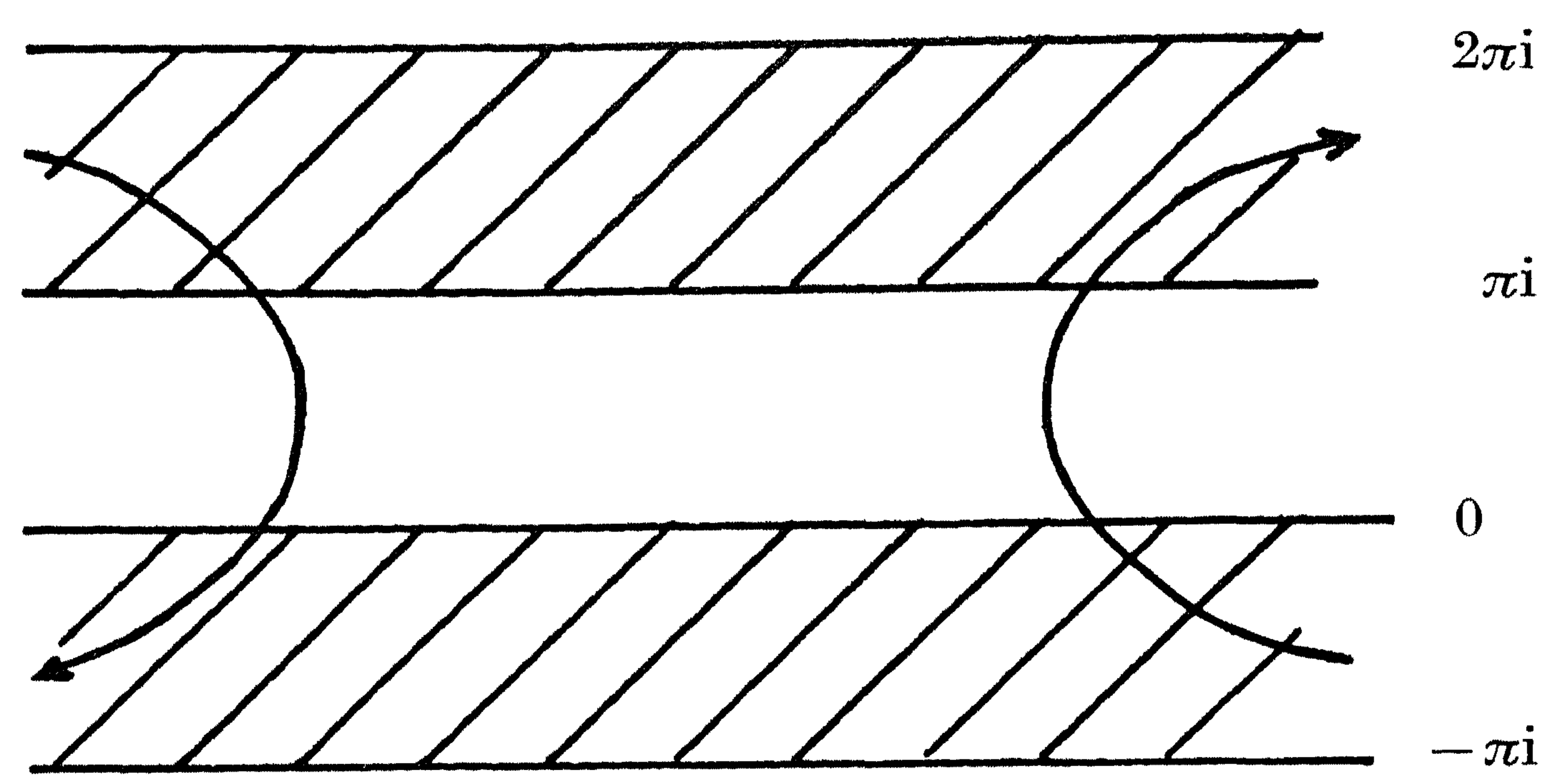

Fig. 3

As shown in the above-mentioned paper the solutions F_m have as regards the origin $r=0$ and infinity $r=\infty$ the character of the Bessel functions $K_m(r)$ whereas the associated solutions $F_m{}^*$ correspond to the Bessel functions $I_m(r)$.

The Fourier-representations (2.19) and (2.21) may be converted to a Laplace-representation in the same way as in the first paper (cf. I 5.19). In order to show again how this can be done we shall consider the case $m=0$ only. Starting from (2.2) we may write

$$(2.22)\quad \begin{cases} F(r,\varphi) = \int\limits_{\mathrm{Im}\, w=\frac{1}{2}\pi+\varphi} \exp(-r\,\mathrm{ch}(w-\mathrm{i}\varphi))\, f_1(w-\tfrac{1}{2}\pi\mathrm{i})\, dw + \\ \qquad + \int\limits_{\mathrm{Im}\, w=-\frac{1}{2}\pi+\varphi} \exp(-r\,\mathrm{ch}(w-\mathrm{i}\varphi))\, f_2(-w-\tfrac{1}{2}\pi\mathrm{i})\, dw. \end{cases}$$

If the two integration lines which are a distance π apart are brought to coincide with the line $\mathrm{Im}\, w=c$ we obtain

$$(2.23)\quad F(r,\varphi) = \int\limits_L \exp(-r\,\mathrm{ch}(w-\mathrm{i}\varphi))\, H(w)\, dw + \text{sum of residues.}$$

By the shifting of the lines $\mathrm{Im}\, w = \pm\frac{1}{2}\pi+\varphi$ a number of poles may be passed. The contribution of their residues is clearly of the form

$$(2.24)\quad \sum_j C_j \exp -r\cos(\varphi-\alpha_j).$$

The function $H(w)$ in the integrand of (2.23) is defined by

$$(2.25)\quad H(w) = f_1(w-\tfrac{1}{2}\pi\mathrm{i}) + f_2(-w-\tfrac{1}{2}\pi\mathrm{i}).$$

Substitution of the expressions (2.5) gives

$$(2.26)\quad H(w) = \frac{e(w-\mathrm{i}\varphi_1-\frac{1}{2}\mathrm{i}\pi,\gamma_2,\beta_2)}{e(w-\mathrm{i}\varphi_2-\frac{1}{2}\mathrm{i}\pi,\gamma_1,\beta_1)} + \frac{e(-w+\mathrm{i}\varphi_1-\frac{1}{2}\mathrm{i}\pi,\gamma_2,\beta_2)}{e(-w+\mathrm{i}\varphi_2-\frac{1}{2}\mathrm{i}\pi,\gamma_1,\beta_1)}.$$

By making use of the functional relation of the Appendix this expression may be reduced to (cf. I 5.21)

$$(2.27)\quad H(w) = \frac{\cos\frac{1}{2}\nu(\beta_1-\beta_2-\gamma_1+\gamma_2)\,\mathrm{ch}\,\nu(w-\frac{1}{2}\mathrm{i}\varphi_1-\frac{1}{2}\mathrm{i}\varphi_2)}{\mathrm{ch}\,\frac{1}{2}\nu(w-\mathrm{i}\varphi_1-\mathrm{i}\gamma_1+\mathrm{i}\beta_1)\,\mathrm{ch}\,\frac{1}{2}\nu(w-\mathrm{i}\varphi_2+\mathrm{i}\gamma_2-\mathrm{i}\beta_2)}\, f_1(w-\tfrac{1}{2}\pi\mathrm{i}).$$

The seemingly complicated function $H(w)$ satisfies the following simple functional relations

$$\{\operatorname{sh}(w-\mathrm{i}\gamma_j)+\operatorname{sh}\mathrm{i}\beta_j\}\,H(\mathrm{i}\varphi_j+w)=\{\operatorname{sh}(w+\mathrm{i}\gamma_j)-\operatorname{sh}\mathrm{i}\beta_j\}\,H(\mathrm{i}\varphi_j-w) \tag{2.28}$$

for $j=1, 2$.

These relations of which I (5.22) is a special case can be derived from (2.25) and the relations (2.3). In fact we have

$$\begin{aligned} H(\mathrm{i}\varphi_j+w) &= f_1(\mathrm{i}\varphi_j+w-\tfrac{1}{2}\mathrm{i}\pi)+f_2(-\mathrm{i}\varphi_j-w-\tfrac{1}{2}\mathrm{i}\pi)= \\ &= \frac{\operatorname{sh}(w+\mathrm{i}\gamma_j)-\operatorname{sh}\mathrm{i}\beta_j}{\operatorname{sh}(w-\mathrm{i}\gamma_j)+\operatorname{sh}\mathrm{i}\beta_j}\{f_2(-\mathrm{i}\varphi_j+w-\tfrac{1}{2}\mathrm{i}\pi)+f_1(\mathrm{i}\varphi_j-w-\tfrac{1}{2}\mathrm{i}\pi)\}= \\ &= \frac{\operatorname{sh}(w+\mathrm{i}\gamma_j)-\operatorname{sh}\mathrm{i}\beta_j}{\operatorname{sh}(w-\mathrm{i}\gamma_j)+\operatorname{sh}\mathrm{i}\beta_j}H(\mathrm{i}\varphi_j-w). \end{aligned}$$

We finally note that $H(w)$ has the following typical poles

$$\begin{cases} w = \mathrm{i}\beta_1(\varphi_1+\gamma_1-\beta_1), \\ w = \mathrm{i}\beta_2(\varphi_2+\gamma_2+\beta_2). \end{cases} \tag{2.29}$$

The other poles can be determined by means of e.g. (2.27).

3. *The G-problem*

The solution of the G-problem may proceed along the lines of II section 2. It will be assumed that the function of Green $G(r, \varphi, r_0, \varphi_0)$ may be represented by the following Fourier integral

$$G(r, \varphi, r_0, \varphi_0) = \tfrac{1}{2}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}r\operatorname{sh}u}\{g_1(u+\mathrm{i}\varphi)+g_2(u-\mathrm{i}\varphi)\}\,du, \tag{3.1}$$

where g_1 and g_2 are sectionally holomorphic functions. The function $g_1(w)$ is holomorphic in the strips $\varphi_1<\operatorname{Im} w<\varphi_0$ and $\varphi_0<\operatorname{Im} w<\varphi_2$. The function $g_2(w)$ is holomorphic in the strips $-\varphi_0<\operatorname{Im} w<-\varphi_1$ and $-\varphi_2<\operatorname{Im} w<-\varphi_0$.

According to **II** (2.12) they have the following discontinuities

$$\begin{cases} g_1(u+\mathrm{i}\varphi)\Big|_{\varphi_0-0}^{\varphi_0+0} = \dfrac{1}{2\pi}\,\mathrm{e}^{\mathrm{i}r_0\operatorname{sh}u}, \\[2ex] g_2(u-\mathrm{i}\varphi)\Big|_{\varphi_0-0}^{\varphi_0+0} = -\dfrac{1}{2\pi}\,\mathrm{e}^{\mathrm{i}r_0\operatorname{sh}u}. \end{cases} \tag{3.2}$$

If we put

$$g_j(w) = f_j(w)\,P_j(w), \qquad j=1, 2, \tag{3.3}$$

where the $f_j(w)$ are defined by (2.5) then the boundary conditions at $\varphi=\varphi_j$ reduce in view of (2.3) to

$$P_1(u+\mathrm{i}\varphi_j) = P_2(u-\mathrm{i}\varphi_j), \tag{3.4}$$

so that we may put

$$P_1(w+\mathrm{i}\varphi_1) = P_2(w-\mathrm{i}\varphi_1) = P(w) \tag{3.5}$$

or

$$P_1(w) = P(w-\mathrm{i}\varphi_1), \quad P_2(w) = P(w+\mathrm{i}\varphi_1), \tag{3.6}$$

where $P(w)$ is a periodic function with the period $2\mathrm{i}\theta$ which is sectionally holomorphic in $-\theta < \mathrm{Im}\, w < \theta$ with discontinuities at $\mathrm{Im}\, w = \pm\, (\varphi_0 - \varphi_1)$.

It follows from (3.2) and (3.3) that

$$P(w) \Big|_{\mathrm{i}(\varphi_0-\varphi_1)-\mathrm{i}0}^{\mathrm{i}(\varphi_0-\varphi_1)+\mathrm{i}0} = \frac{\mathrm{e}^{\mathrm{i}r_0 \,\mathrm{sh}\, u}}{2\pi f_1(u+\mathrm{i}\varphi_0)}, \tag{3.7}$$

and

$$P(w) \Big|_{-\mathrm{i}(\varphi_0-\varphi_1)-\mathrm{i}0}^{-\mathrm{i}(\varphi_0-\varphi_1)+\mathrm{i}0} = \frac{\mathrm{e}^{\mathrm{i}r_0 \,\mathrm{sh}\, u}}{2\pi f_2(u-\mathrm{i}\varphi_0)}. \tag{3.8}$$

A solution of (3.7) and (3.8) can be obtained in the form of two Cauchy-integrals

$$P(w) = \frac{\nu}{4\pi\mathrm{i}} \int_{l_1} \frac{\exp \mathrm{i}r_0 \,\mathrm{sh}\,(w_0 - \mathrm{i}\varphi_0 + \mathrm{i}\varphi_1)}{2\pi f_1(w_0+\mathrm{i}\varphi_1)} \,\mathrm{cth}\, \tfrac{1}{2}\nu(w_0 - w)\, dw_0 +$$
$$+ \frac{\nu}{4\pi\mathrm{i}} \int_{l_2} \frac{\exp \mathrm{i}r_0 \,\mathrm{sh}\,(w_0 + \mathrm{i}\varphi_0 - \mathrm{i}\varphi_1)}{2\pi f_2(w_0-\mathrm{i}\varphi_1)} \,\mathrm{cth}\, \tfrac{1}{2}\nu(w_0 - w)\, dw_0,$$

where the lines of integration l_1 and l_2 are determined by $\mathrm{Im}\, w_0 = \varphi_0 - \varphi_1$ for l_1 and $\mathrm{Im}\, w_0 = -\varphi_0 + \varphi_1$ for l_2 with in both cases $\mathrm{Re}\, w_0$ running from $-\infty$ to $+\infty$.

Substitution of (3.9) in (3.6) and (3.1) gives eventually

$$G(r,\varphi,r_0,\varphi_0) = \frac{\nu}{16\pi^2\mathrm{i}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(-\mathrm{i}r \,\mathrm{sh}\, u + \mathrm{i}r_0 \,\mathrm{sh}\, u_0)\, g(u,u_0,\varphi,\varphi_0)\, du\, du_0, \tag{3.10}$$

where

$$g(u,\, u_0,\, \varphi,\, \varphi_0) = \frac{f_1(v)}{f_1(v_0)} \,\mathrm{cth}\, \tfrac{1}{2}\nu(v_0 - v) + \frac{f_1(v)}{f_2(v_0{}^*)} \,\mathrm{cth}\, \tfrac{1}{2}\nu(v_0{}^* - v) +$$
$$+ \frac{f_2(v^*)}{f_1(v_0)} \,\mathrm{cth}\, \tfrac{1}{2}\nu(v_0 - v^*) + \frac{f_2(v^*)}{f_2(v_0{}^*)} \,\mathrm{cth}\, \tfrac{1}{2}\nu(v_0{}^* - v^*),$$

with $v = u + \mathrm{i}\varphi$ and $v_0 = u_0 + \mathrm{i}\varphi_0$.

The convergence of (3.10) can be effected in a variety of ways e.g. by shifting the lines of integration a little upward or downward or by subtraction of an appropriate solution of the homogeneous problem. Anyhow the form (3.10) presents no more difficulties than the corresponding solution of the simpler case which has been treated in the second paper.

4. *The G-problem, second method*

Following the method of II section 4 we may try to represent the Green's function G in the following way by a Laplace integral (cf. II 4.5)

$$G(r,\varphi,r_0,\varphi_0) = G_0(r,\varphi,r_0,\varphi_0) + \frac{1}{4\pi}\int_{-\infty+ic}^{\infty+ic} \exp\{-r\,\mathrm{ch}\,(w-\mathrm{i}\varphi)\}\,g(w)\,dw, \tag{4.1}$$

where

$$G_0(r,\varphi,r_0,\varphi_0) = \frac{1}{2\pi}K_0(\sqrt{r^2+r_0^2-2rr_0\cos(\varphi-\varphi_0)}). \tag{4.2}$$

We shall provisionally assume that $\frac{1}{2}\pi<\theta<\pi$.

From

$$G_0(r,\varphi,r_0,\varphi_0) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \exp\{-\mathrm{i}r\,\mathrm{sh}\,(u+\varepsilon\mathrm{i}\varphi)+\mathrm{i}r_0\,\mathrm{sh}\,(u+\varepsilon\mathrm{i}\varphi_0)\}\,du \tag{4.3}$$

with either $\varepsilon=1$ or $\varepsilon=-1$ it follows that by applying the boundary conditions at $\varphi=\varphi_j$ we have for $\varphi=\varphi_1$

$$\left\{\begin{aligned} &\left(\cos\gamma_1\frac{1}{r}\frac{\partial}{\partial\varphi}-\sin\gamma_1\frac{\partial}{\partial r}-\sin\beta_1\right)G_0 = \\ &= \frac{1}{4\pi}\int_{-\infty}^{\infty}\exp\{-\mathrm{i}r\,\mathrm{sh}\,u+\mathrm{i}r_0\,\mathrm{sh}\,(u+\mathrm{i}\varphi_0-\mathrm{i}\varphi_1)\}\{\mathrm{ch}\,(u+\mathrm{i}\gamma_1)-\sin\beta_1\}\,du,\end{aligned}\right. \tag{4.4}$$

and for $\varphi=\varphi_2$

$$\left\{\begin{aligned} &\left(\cos\gamma_2\frac{1}{r}\frac{\partial}{\partial\varphi}-\sin\gamma_2\frac{\partial}{\partial r}-\sin\beta_2\right)G_0 = \\ &= -\frac{1}{4\pi}\int_{-\infty}^{\infty}\exp\{\mathrm{i}r\,\mathrm{sh}\,u-\mathrm{i}r_0\,\mathrm{sh}\,(u+\mathrm{i}\varphi_0-\mathrm{i}\varphi_2)\}\{\mathrm{ch}\,(u+\mathrm{i}\gamma_2)-\sin\beta_2\}\,du.\end{aligned}\right. \tag{4.5}$$

Then reasoning along the same lines as in II section 4 the boundary conditions lead to the requirement that there exist holomorphic functions $g_j(w)$ of which $g_1(w)$ is holomorphic in the *lower* strip $-\pi<\mathrm{Im}\,w<0$ and $g_2(w)$ holomorphic in the *upper* strip $0<\mathrm{Im}\,w<\pi$ [1]) in such a way that

$$\left\{\begin{aligned} g_1(-\tfrac{1}{2}\pi\mathrm{i}+w) &= g_1(-\tfrac{1}{2}\pi\mathrm{i}-w),\\ g_2(\tfrac{1}{2}\pi\mathrm{i}+w) &= g_2(\tfrac{1}{2}\pi\mathrm{i}-w),\end{aligned}\right. \tag{4.6}$$

$$\left\{\begin{aligned} g_1(w)\,\mathrm{ch}\,w &= \{\mathrm{ch}\,(w-\mathrm{i}\gamma_1)+\sin\beta_1\}\,g(w+\mathrm{i}\varphi_1+\tfrac{1}{2}\mathrm{i}\pi)+\\ &\quad-\{\mathrm{ch}\,(w+\mathrm{i}\gamma_1)-\sin\beta_1\}\exp\{\mathrm{i}r_0\,\mathrm{sh}\,(w+\mathrm{i}\varphi_0-\mathrm{i}\varphi_1)\},\end{aligned}\right. \tag{4.7}$$

and

$$\left\{\begin{aligned} g_2(w)\,\mathrm{ch}\,w &= \{\mathrm{ch}\,(w-\mathrm{i}\gamma_2)-\sin\beta_2\}\,g(w+\mathrm{i}\varphi_2-\tfrac{1}{2}\mathrm{i}\pi)+\\ &\quad-\{\mathrm{ch}\,(w+\mathrm{i}\gamma_2)+\sin\beta_2\}\exp\{-\mathrm{i}r_0\,\mathrm{sh}\,(w+\mathrm{i}\varphi_0-\mathrm{i}\varphi_2)\}.\end{aligned}\right. \tag{4.8}$$

---

[1]) In the corresponding passage in II p. 362 the two strips have been interchanged erroneously.

Elimination of $g(w)$ from these expressions yields in accordance with II (4.14)

$$h_2(w)\,g_2(w-\mathrm{i}\varphi_2+\tfrac{1}{2}\mathrm{i}\pi)-h_1(w)\,g_1(w-\mathrm{i}\varphi_1-\tfrac{1}{2}\mathrm{i}\pi) = k_1(w)-k_2(w). \tag{4.9}$$

valid in the strip

$$\varphi_2-\tfrac{1}{2}\pi<\operatorname{Im} w<\varphi_1+\tfrac{1}{2}\pi, \tag{4.10}$$

and where for $j=1$ and $j=2$

$$h_j(w) \stackrel{\text{def}}{=} \frac{\operatorname{sh}(w-\mathrm{i}\varphi_j)}{\operatorname{sh}(w-\mathrm{i}\varphi_j-\mathrm{i}\gamma_j)+\operatorname{sh}\mathrm{i}\beta_j}, \tag{4.11}$$

and

$$k_j(w) \stackrel{\text{def}}{=} \frac{\operatorname{sh}(w-\mathrm{i}\varphi_j+\mathrm{i}\gamma_j)-\operatorname{sh}\mathrm{i}\beta_j}{\operatorname{sh}(w-\mathrm{i}\varphi_j-\mathrm{i}\gamma_j)+\operatorname{sh}\mathrm{i}\beta_j}\exp\{r_0\operatorname{ch}(w-2\mathrm{i}\varphi_j+\mathrm{i}\varphi_0)\}. \tag{4.12}$$

The solution of the generalized Hilbert problem (4.9) involves the factorisation

$$h_1(w)\,H_1(w) = h_2(w)\,H_2(w) \tag{4.13}$$

where $H_j(w)$ is holomorphic in the strip $\varphi_j-\frac{1}{2}\pi<\operatorname{Im} w<\varphi_j+\frac{1}{2}\pi$ and symmetric with respect to $\mathrm{i}\varphi_j$.

If we define $H(w)=h_j(w)H_j(w)$, where according to (4.13) j is either 1 or 2, from the symmetry relations of $H_j(w)$ the following functional relations of $H(w)$ may be derived.

$$\frac{H(\mathrm{i}\varphi_j+w)}{H(\mathrm{i}\varphi_j-w)} = \frac{h_j(\mathrm{i}\varphi_j+w)}{h_j(\mathrm{i}\varphi_j-w)} = \frac{\operatorname{sh}(w+\mathrm{i}\gamma_j)-\operatorname{sh}\mathrm{i}\beta_j}{\operatorname{sh}(w-\mathrm{i}\gamma_j)+\operatorname{sh}\mathrm{i}\beta_j}. \tag{4.14}$$

Since these are exactly the functional relations (2.28) we may identify the function $H(w)$ defined by (4.13) with the function $H(w)$ which has been studied in section 2 and which was defined by (2.25).

Hence the factorisation (4.13) is solved by

$$H_j(w) = H(w)/h_j(w) \tag{4.15}$$

under the proviso that $H_j(w)$ is free from poles in the strip

$$\varphi_j-\tfrac{1}{2}\pi<\operatorname{Im} w<\varphi_j+\tfrac{1}{2}\pi.$$

As in II section 4 it follows that this is true only when the following two inequalities hold (cf. II 4.1)

$$\operatorname{Re}\gamma_1<\theta+\operatorname{Re}\beta_1-\tfrac{1}{2}\pi,\quad \operatorname{Re}\gamma_2>-\theta-\operatorname{Re}\beta_2+\tfrac{1}{2}\pi \tag{4.16}$$

The factorisation problem (4.13) being solved the problem (4.9) can be replaced by the simpler one

$$\frac{g_2(w-\mathrm{i}\varphi_2+\frac{1}{2}\mathrm{i}\pi)}{H_2(w)}-\frac{g_1(w-\mathrm{i}\varphi_1-\frac{1}{2}\mathrm{i}\pi)}{H_1(w)} = \frac{k_1(w)-k_2(w)}{H(w)}. \tag{4.17}$$

This problem is identical with the one which has been studied and solved in II section 4 (cf. II 4.19 sqq.). Therefore the Green's function can be represented by II (4.25) where of course $k_j(w)$ and $H(w)$ are more general. In order to avoid continuous reference to the quoted paper the solution will be reproduced here

$$
(4.18)\quad \left\{
\begin{aligned}
&2\pi G(r,\,\varphi,\,r_0,\,\varphi_0) = K_0(\sqrt{r^2+r_0{}^2-2rr_0\cos(\varphi-\varphi_0)})+\\
&+\tfrac{1}{2}\int_{-\infty}^{\infty} \mathrm{e}^{-r\,\mathrm{ch}\,(w-\mathrm{i}\varphi)}\,k_1(w)\,dw+\frac{\nu}{4\pi\mathrm{i}}\int_{-\infty+\mathrm{i}c}^{\infty+\mathrm{i}c} \mathrm{e}^{-r\,\mathrm{ch}\,(w-\mathrm{i}\varphi)}\,H(w)\,dw\,\cdot\\
&\cdot\int_{L_1} \frac{k_1(w_0)-k_2(w_0)}{H(w_0)}\,\frac{\mathrm{sh}\,\nu(w-\mathrm{i}\varphi_1)}{\mathrm{ch}\,\nu(w_0-\mathrm{i}\varphi_1)-\mathrm{ch}\,\nu(w-\mathrm{i}\varphi_1)}\,dw_0
\end{aligned}
\right.
$$

where the line of integration L_1 is determined by $\mathrm{Im}\,w<\mathrm{Im}\,w_0<\varphi_1+\frac{1}{2}\pi$ as shown in II figure 2. Also in the general case studied here the expression (4.18) can be reduced to the form II (4.28) i.e.

$$
(4.19)\quad \left\{
\begin{aligned}
&2\pi G(r,\,\varphi,\,r_0,\,\varphi_0) =\\
&=\frac{\nu}{4\pi\mathrm{i}}\int_{-\infty+\mathrm{i}c}^{\infty+\mathrm{i}c} \exp\{-r\,\mathrm{ch}\,(w-\mathrm{i}\varphi)\}\,H(w)\,dw\int_L \exp\{r_0\,\mathrm{ch}\,(w_0-\mathrm{i}\varphi_0)\}\,\cdot\\
&\cdot\,H^{-1}(w_0)\,\frac{\mathrm{sh}\,\nu(w_0-\mathrm{i}\varphi_1)}{\mathrm{ch}\,\nu(w_0-\mathrm{i}\varphi_1)-\mathrm{ch}\,\nu(w-\mathrm{i}\varphi_1)}\,dw_0,
\end{aligned}
\right.
$$

where L is a contour as shown in figure 3.

We finally note that the expansion II (4.35) also holds good in the general case. The discussion of (4.19) does not lead to essentially new features and will therefore be omitted here.

5. *Appendix*

We shall prove the following functional relations of the function $e(z,\gamma,\beta)$ defined by (2.11) for the case (2.8) and by (2.13) for all values of z, γ and β.

$$
(5.1)\qquad \frac{e(\frac{1}{2}\pi\mathrm{i}+z,\,\gamma,\,\beta)}{e(\frac{1}{2}\pi\mathrm{i}-z,\,\gamma,\,\beta)} = \frac{\mathrm{ch}\,\frac{1}{2}\nu(z+\mathrm{i}\gamma+\mathrm{i}\beta)}{\mathrm{ch}\,\frac{1}{2}\nu(z-\mathrm{i}\gamma-\mathrm{i}\beta)}.
$$

and

$$
(5.2)\qquad \frac{e(-\frac{1}{2}\pi\mathrm{i}+z,\,\gamma,\,\beta)}{e(-\frac{1}{2}\pi\mathrm{i}-z,\,\gamma,\,\beta)} = \frac{\mathrm{ch}\,\frac{1}{2}\nu(z-\mathrm{i}\gamma+\mathrm{i}\beta)}{\mathrm{ch}\,\frac{1}{2}\nu(z+\mathrm{i}\gamma-\mathrm{i}\beta)}.
$$

These equations are a generalization of I (4.13). They can be proved by taking the logarithmic derivative of both sides and using (2.6). Considering (5.1) we have

$$
\frac{d}{dz}\ln\frac{e(\frac{1}{2}\pi\mathrm{i}+z,\,\gamma,\,\beta)}{e(\frac{1}{2}\pi\mathrm{i}-z,\,\gamma,\,\beta)} = 2\int_{-\infty}^{\infty}\cos zt\;\mathrm{e}^{\frac{1}{2}\pi t}\,\psi(t)\,dt.
$$

Substitution of (2.9) in the right-hand side gives in view of III (6.2)

$$
\begin{aligned}
\mathrm{i}\int_{-\infty}^{\infty}\frac{\cos zt\,\mathrm{sh}\,(\gamma+\beta)}{\mathrm{sh}\,\theta t}\,dt &= \tfrac{1}{2}\nu\mathrm{i}\{\mathrm{tg}\,\tfrac{1}{2}\nu(\mathrm{i}z+\gamma+\beta)+\mathrm{tg}\,\tfrac{1}{2}\nu(-\mathrm{i}z+\gamma+\beta)\} =\\
&= \frac{d}{dz}\ln\frac{\cos\frac{1}{2}\nu(-\mathrm{i}z+\gamma+\beta)}{\cos\frac{1}{2}\nu(\mathrm{i}z+\gamma+\beta)},
\end{aligned}
$$

from which (5.1) easily follows, at least apart from a multiplicative constant. However, (5.1) obviously is correct for $z=0$ so that it is correct for all z. The proof of (5.2) is similar.

Next we prove the functional relation

$$\frac{e(z-\mathrm{i}\theta,\gamma,\beta)}{e(z,\gamma-\theta,\beta)}=C\{\operatorname{ch}(z-\mathrm{i}\gamma)+\sin\beta\}, \tag{5.3}$$

which is a generalization of I (4.14).

For the proof of (5.3) we may use the knowledge of the poles and zeros of $e(z,\gamma,\beta)$ together with its asymptotic expansion.

Inspection of (2.13) shows that $e(z,\gamma,\beta)$ has poles at

$$\begin{cases} -\mathrm{i}\beta\pm\mathrm{i}\{(4n+3)\,\tfrac{1}{2}\pi+(2m+1)\,\theta+\gamma\}, \\ \mathrm{i}\beta\pm\mathrm{i}\{(4n+1)\,\tfrac{1}{2}\pi+(2m+1)\,\theta+\gamma\}, \end{cases} \tag{5.4}$$

and zeros at

$$\begin{cases} -\mathrm{i}\beta\pm\mathrm{i}\{(4n+1)\,\tfrac{1}{2}\pi+(2m+1)\,\theta-\gamma\}, \\ \mathrm{i}\beta\pm\mathrm{i}\{(4n+3)\,\tfrac{1}{2}\pi+(2m+1)\,\theta-\gamma\}, \end{cases} \tag{5.5}$$

where $m, n=0, 1, 2, \ldots$.

Therefore the left-hand side of (5.3) has no poles and only a set of zeros which are those of the right-hand side. Since both sides have the same asymptotic behaviour as $\operatorname{Re} z\to\pm\infty$ the equality holds with some constant C.

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